

# Controlling stochastic sensitivity by the dynamic regulators

Cite as: AIP Conference Proceedings **1895**, 050002 (2017); <https://doi.org/10.1063/1.5007374>  
Published Online: 12 October 2017

I. Bashkirtseva



View Online



Export Citation

## ARTICLES YOU MAY BE INTERESTED IN

[Analysis of stochastic phenomena in Ricker-type population model with delay](#)

AIP Conference Proceedings **1895**, 050003 (2017); <https://doi.org/10.1063/1.5007375>

[Modeling and stochastic analysis of dynamic mechanisms of the perception](#)

AIP Conference Proceedings **1895**, 050008 (2017); <https://doi.org/10.1063/1.5007380>

[Analysis of the stochastically forced invariant manifolds of dynamic systems](#)

AIP Conference Proceedings **1895**, 050010 (2017); <https://doi.org/10.1063/1.5007382>

**AIP** | Conference Proceedings

Get **30% off** all  
print proceedings!

Enter Promotion Code **PDF30** at checkout



# Controlling Stochastic Sensitivity by the Dynamic Regulators

I. Bashkirtseva

*Ural Federal University, 51 Lenin str., Ekaterinburg 620000, Russia*

*irina.bashkirtseva@urfu.ru*

**Abstract.** A problem of the control of stochastically forced equilibria in nonlinear dynamic systems with incomplete information is considered. Our approach is based on the idea of the synthesis of the required stochastic sensitivity for the equilibrium. We consider an important, from engineering point of view, case when the system states are observed partially, and these observations contain random noises. To solve this problem, a dynamic regulator consisting of the feedback and filter is constructed. Mathematically, the problem of the synthesis of the assigned stochastic sensitivity is reduced to the solution of the matrix equations for coefficients of the regulator. Problems of the controllability and attainability are discussed. The effectiveness of the proposed approach is demonstrated on the examples.

## INTRODUCTION

In the study of control problems, the incomplete information refers to dynamical systems in which system states are known limitedly because of the difficulties in the observation and explicit quantification. The standard type of the incomplete information is connected with the fact that not all coordinates of the system states are observable, and data contain random errors.

For stochastic dynamic systems, the control theory is actively developing, and attracts attention of both mathematicians and engineers (see [1, 2, 3, 4, 5]). The control problems for linear dynamic systems have a deep mathematical theory. Here, Linear-Quadratic-Gaussian problem [6] and separation principle that combines Kalman filter (linear-quadratic estimator) and a linear-quadratic regulator [7, 8] are well known. Recently, the filtration and control of nonlinear stochastic systems is a problem of a high importance, especially in the modern engineering [9, 10].

Random disturbances in nonlinear systems can generate various complex stochastic phenomena [11, 12, 13]. In nonlinear systems, the full probabilistic description of the stochastic dynamics is given by the Kolmogorov-Fokker-Planck equation. However, this equation can be solved analytically only in very specific cases, so the numerical simulation is still a main tool of the nonlinear stochastic systems analysis. But for solving control problems, this numerical approach is not applicable. So, the asymptotical approximations are the constructive alternative. Here, the quasipotential method [14] and the stochastic sensitivity function (SSF) technique [15, 16, 17] were successfully used for the solution of control problems [18, 19].

The main idea of control on the base of SSF is to decrease a level of the sensitivity of attractors (equilibria, cycles) to noise. In systems with control, one can change a level of the stochastic sensitivity by an appropriate regulator. This approach was elaborated for controlling the stochastic sensitivity of the equilibria, and applied to the stabilization of the operation modes of engineering devices. In the case of the complete information, the theory of the synthesis of the stochastic sensitivity for continuous and discrete systems is presented in [20].

In practice, the available data on the current system states are usually restricted and contain random noise or errors [21]. The problem of the synthesis of the stochastic sensitivity for systems with incomplete information by static feedback regulators was considered in [22, 23]. A novelty of the present paper is that for the first time the problem of the synthesis of the stochastic sensitivity for systems with the incomplete information is solved by the help of dynamic regulators.

## STOCHASTIC SENSITIVITY OF THE EQUILIBRIUM

Consider a general nonlinear stochastic system

$$\dot{x} = f(x) + \varepsilon \sigma(x) \xi(t), \quad (1)$$

where  $x$  is an  $n$ -dimensional vector,  $f(x)$  is an  $n$ -dimensional vector-function,  $\xi(t)$  is an  $m$ -dimensional white Gaussian noise with parameters  $E\xi(t) = 0$ ,  $E\xi(t)\xi^\top(\tau) = \delta(t - \tau)I$ ,  $I$  is the identity matrix,  $\sigma(x)$  is an  $(n \times m)$ -matrix-function, and  $\varepsilon$  is a scalar parameter of the noise intensity.

Let  $\bar{x}$  be an exponentially stable equilibrium of the unforced deterministic system (1) (with  $\varepsilon = 0$ ).

Consider the asymptotics

$$z(t) = \lim_{\varepsilon \rightarrow 0} \frac{x^\varepsilon(t) - \bar{x}}{\varepsilon} \quad (2)$$

of the deviations of solutions  $x^\varepsilon(t)$  of system (1) from the equilibrium  $\bar{x}$ . The variable  $z(t)$  is governed by the following stochastic system

$$\dot{z} = Fz + S\xi, \quad F = \frac{\partial f}{\partial x}(\bar{x}), \quad S = \sigma(\bar{x}). \quad (3)$$

For the covariance matrix  $V(t) = \text{cov}(z(t), z(t))$  we have the linear deterministic equation:

$$\dot{V} = FV + VF^\top + SS^\top.$$

For the stable equilibrium  $\bar{x}$ , this equation has a unique stationary solution  $W$  governed by the following matrix algebraic equation:

$$FW + WF^\top + SS^\top = 0. \quad (4)$$

The stochastic sensitivity matrix  $W$  is a simple quantitative characteristic of a response of the nonlinear system (1) to weak noise near the equilibrium  $\bar{x}$ . For the stationary distributed states  $\bar{x}^\varepsilon(t)$  of system (1) it holds that  $\text{cov}(\bar{x}^\varepsilon(t), \bar{x}^\varepsilon(t)) \approx \varepsilon^2 W$ .

## CONTROL OF STOCHASTIC SENSITIVITY

Consider a nonlinear controlled stochastic system

$$\dot{x} = f(x, u) + \varepsilon \sigma(x) \xi(t) \quad (5)$$

where  $u$  is an  $r$ -dimensional control input.

It is assumed that the corresponding deterministic system (5) (with  $\varepsilon = 0$  and  $u = 0$ ) has an equilibrium  $\bar{x}$  whose stability is not supposed.

It is known that the full information on the current state  $x(t)$  is frequently inaccessible. In the present paper, we consider a case that the measurement vector  $y(t)$  contains the random noise:

$$y = g(x) + \varepsilon \varphi(x) \eta(t). \quad (6)$$

Here,  $y$  is an  $p$ -dimensional vector of the measurement,  $g(x)$  is an  $l$ -dimensional vector-function,  $\varphi(x)$  is  $p \times q$  matrix function, and  $\eta(t)$  is the standard white Gaussian  $q$ -vector noise, uncorrelated with  $\xi(t)$ , satisfying  $E\eta(t) = 0$ ,  $E\eta(t)\eta^\top(\tau) = \delta(t - \tau)I$ .

In the present paper, we will use a dynamic regulator composed of the feedback

$$u = -K(\hat{x} - \bar{x}) \quad (7)$$

and the filter

$$\dot{\hat{x}} = f(\hat{x}, u) + L(y - g(\hat{x})). \quad (8)$$

Here,  $\hat{x}$  is an estimation of the unknown state  $x$ , and  $K$  is a constant  $r \times n$ -matrix,  $L$  is a constant  $n \times p$ -matrix. So, stabilizing abilities of this regulator are defined by the appropriate choice of the pair of matrices  $K$  and  $L$ .

For states  $x$  and  $\hat{x}$  of system (5), (6) with the regulator (7), (8), one can write the following closed system:

$$\begin{aligned}\dot{x} &= f(x, -K(\hat{x} - \bar{x})) + \varepsilon \sigma(x) \xi(t) \\ \dot{\hat{x}} &= f(\hat{x}, -K(\hat{x} - \bar{x})) + L(g(x) - g(\hat{x})) + \varepsilon L \varphi(x) \eta(t).\end{aligned}\tag{9}$$

The deterministic system (9) (with  $\varepsilon = 0$ ) has the equilibrium  $[\bar{x}, \bar{x}]^\top$ .

For the asymptotics

$$z = \lim_{\varepsilon \rightarrow 0} \frac{x^\varepsilon - \bar{x}}{\varepsilon}, \quad \hat{z} = \lim_{\varepsilon \rightarrow 0} \frac{\hat{x}^\varepsilon - \bar{x}}{\varepsilon}$$

of deviations of system (9) states  $x^\varepsilon$ ,  $\hat{x}^\varepsilon$  from the equilibrium, the following system can be written:

$$\begin{aligned}\dot{z} &= Az - BK\hat{z} + S\xi(t) \\ \dot{\hat{z}} &= A\hat{z} - BK\hat{z} + LC(z - \hat{z}) + L\Phi\eta(t).\end{aligned}\tag{10}$$

Here,

$$A = \frac{\partial f}{\partial x}(\bar{x}, 0), \quad B = \frac{\partial f}{\partial u}(\bar{x}, 0), \quad C = \frac{\partial g}{\partial x}(\bar{x}), \quad S = \sigma(\bar{x}), \quad \Phi = \varphi(\bar{x}).$$

Consider the sets of matrices  $K$  and  $L$  that provide an exponential stability of the equilibrium  $[\bar{x}, \bar{x}]^\top$  of the deterministic system (9) with  $\varepsilon = 0$ :

$$\mathbf{K} = \{K \mid \Re \lambda_i(A - BK) < 0\}, \quad \mathbf{L} = \{L \mid \Re \lambda_i(A - LC) < 0\},$$

where  $\lambda_i(F)$  are the eigenvalues of the matrix  $F$ . Suppose that the pair  $(A, B)$  is stabilizable, and the pair  $(A, C)$  is detectable [24]. This means that the sets  $\mathbf{K}$ ,  $\mathbf{A}$  are not empty.

Let us rewrite the system (10) in the following form:

$$\begin{bmatrix} \dot{z} \\ \dot{\hat{z}} \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A - BK - LC \end{bmatrix} \begin{bmatrix} z \\ \hat{z} \end{bmatrix} + \begin{bmatrix} S & 0 \\ 0 & L\Phi \end{bmatrix} \begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix}.\tag{11}$$

For the variable  $\tilde{z} = \hat{z} - z$ , one can write the following separate subsystem:

$$\dot{\tilde{z}} = (A - LC)\tilde{z} + L\Phi\eta(t) - S\xi(t).$$

For the variables  $z$ ,  $\tilde{z}$ , it holds that

$$\begin{bmatrix} \dot{z} \\ \dot{\tilde{z}} \end{bmatrix} = \begin{bmatrix} A - BK & -BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} z \\ \tilde{z} \end{bmatrix} + \begin{bmatrix} S & 0 \\ -S & L\Phi \end{bmatrix} \begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix}.\tag{12}$$

For any  $K \in \mathbf{K}$ ,  $L \in \mathbf{L}$ , the stochastic sensitivity matrix

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$$

consists of the blocks

$$W_{11} = Ezz^\top, \quad W_{12} = Ez\tilde{z}^\top, \quad W_{21} = E\tilde{z}z^\top, \quad W_{22} = E\tilde{z}\tilde{z}^\top.$$

This matrix is a unique solution of the following matrix algebraic equation

$$\begin{aligned}& \begin{bmatrix} A - BK & -BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} + \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \begin{bmatrix} (A - BK)^\top & 0 \\ -K^\top B^\top & (A - LC)^\top \end{bmatrix} \\ & + \begin{bmatrix} S & 0 \\ -S & L\Phi \end{bmatrix} \begin{bmatrix} S^\top & -S^\top \\ 0 & \Phi^\top L^\top \end{bmatrix} = 0.\end{aligned}\tag{13}$$

Taking into account  $W_{21} = W_{12}^\top$ , one can rewrite system (13) as follows:

$$\begin{aligned}(A - BK)W_{11} + W_{11}(A - BK)^\top - BKW_{12}^\top - W_{12}K^\top B^\top + SS^\top &= 0 \\ (A - BK)W_{12} - BKW_{22} + W_{12}(A - LC)^\top - SS^\top &= 0 \\ (A - LC)W_{22} + W_{22}(A - LC)^\top + SS^\top + L\Phi\Phi^\top L^\top &= 0.\end{aligned}\quad (14)$$

Here, the following problem can be considered.

Let  $\mathbf{M}$  be a set of symmetric and positive definite  $(n \times n)$ -matrices. For any  $K \in \mathbf{K}$ ,  $L \in \mathbf{L}$ , the regulator (7),(8) forms for the equilibrium  $\bar{x}$  the stochastic sensitivity matrix  $W_{11}(K, L)$  which is an  $n \times n$ -block of the  $(2n) \times (2n)$ -matrix solution of system (14).

**Problem of stochastic sensitivity synthesis**

For an assigned matrix  $W_{11} \in \mathbf{M}$ , find the matrices  $K \in \mathbf{K}$ ,  $L \in \mathbf{L}$ , such that the equality  $W_{11}(K, L) = W_{11}$  holds.

In the general case, a question of the attainability of the assigned fixed matrix, and the description of the set of all attainable matrices is still open.

Choosing the matrix  $L$  in the filter (8), one can follow the Kalman–Bucy theory of the optimal filtration:

$$L_* = W_{22}C^\top (\Phi\Phi^\top)^{-1}, \quad (15)$$

where

$$AW_{22} + W_{22}A^\top - W_{22}C^\top (\Phi\Phi^\top)^{-1} CW_{22} + SS^\top = 0.$$

## SYNTHESIS OF STOCHASTIC SENSITIVITY FOR ONE-DIMENSIONAL SYSTEMS

Let in system (5), (6), variables  $x$ ,  $y$ ,  $u$ , and noises  $\xi$  and  $\eta$  be one-dimensional:  $n = m = r = p = q = 1$ . In this case, the feedback (7) and the filter (8) can be written as

$$u = -k(\hat{x} - \bar{x}), \quad \dot{\hat{x}} = f(\hat{x}, u) + l(y - g(\hat{x})). \quad (16)$$

Elements of the corresponding stochastic sensitivity  $2 \times 2$ -matrix

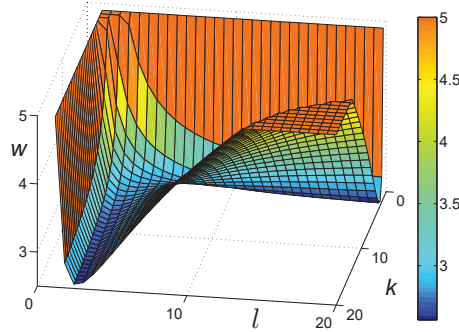


FIGURE 1. Plot of the stochastic sensitivity function  $w = w_{11}(k, l)$  for  $a = b = c = \varphi = s = 1$

$$W = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$$

satisfy the system

$$\begin{aligned}2(a - bk)w_{11} - 2bkw_{12}^\top + s^2 &= 0 \\ (2a - bk - lc)w_{12} - bk w_{22} - s^2 &= 0 \\ 2(a - lc)w_{22} + s^2 + l^2 \varphi^2 &= 0.\end{aligned}\quad (17)$$

Here,

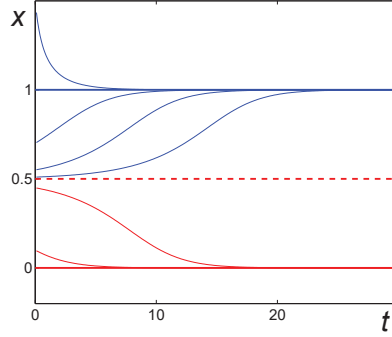
$$a = f'_x(\bar{x}, 0), \quad b = f'_u(\bar{x}, 0), \quad c = g'(\bar{x}), \quad s = \sigma(\bar{x}), \quad \varphi = \varphi(\bar{x}).$$

In this one-dimensional case, we have  $\mathbf{K} = \{k \mid a - bk < 0\}$ , and  $\mathbf{L} = \{l \mid a - lc < 0\}$ .

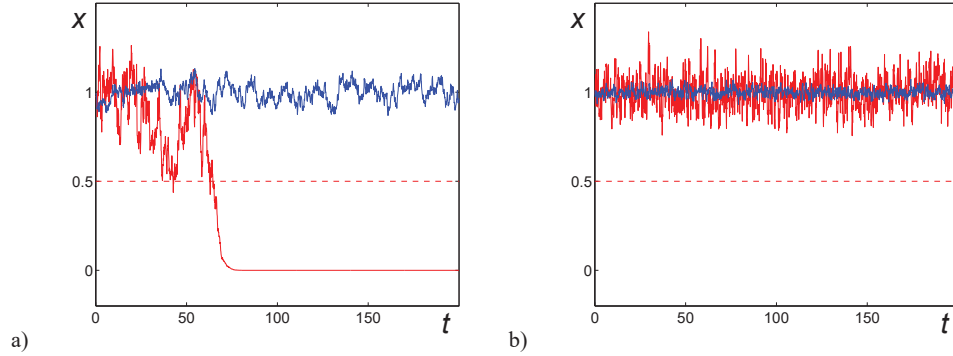
For any  $k \in \mathbf{K}$ ,  $l \in \mathbf{L}$ , the system (17) has an explicit solution:

$$w_{22}(l) = \frac{s^2 + l^2 \varphi^2}{2(lc - a)}, \quad w_{12}(k, l) = \frac{bk w_{22}(l) + s^2}{2a - bk - lc}, \quad w_{11}(k, l) = \frac{s^2 - 2bk w_{12}(k, l)}{2(bk - a)}.$$

In system (5), (6) with regulator (16), the stochastic sensitivity of the equilibrium  $\bar{x}$  is defined by the function  $w = w_{11}(k, l)$ . In Figure 1, the plot of the function  $w = w_{11}(k, l)$  is presented for  $a = b = c = \varphi = s = 1$ .



**FIGURE 2.** Trajectories of the deterministic system



**FIGURE 3.** Time series for  $p = 0.5$  and  $\varepsilon = 0.05$  (blue),  $\varepsilon = 0.2$  (red): a) without control; b) with control

The attainability analysis in the problem of the synthesis of the stochastic sensitivity is reduced to the inverse problem of the solvability of system (17) with respect to variables  $k$  and  $l$  when  $w_{11}$ ,  $w_{12}$ ,  $w_{22}$  are assigned.

## EXAMPLE

Consider the one-dimensional deterministic population model with Allee effect [25, 26]

$$\dot{x} = x(x - p)(1 - x),$$

where the parameter  $p$  ( $0 < p < 1$ ) is the Allee threshold. This model has two stable equilibria  $\bar{x}_0 = 0$  and  $\bar{x}_2 = 1$  separated by the unstable equilibrium  $\bar{x}_1 = p$ . If the initial state  $x_0 < \bar{x}_1$ , then the population is extinct:  $\lim_{t \rightarrow +\infty} x(t) = 0$ .

If  $x_0 > \bar{x}_1$ , then the size of the population stabilizes to the non-trivial equilibrium  $\bar{x}_2$ :  $\lim_{t \rightarrow +\infty} x(t) = \bar{x}_2$ . In Figure 2, these two variants are shown for  $p = 0.5$ .

Further, consider the corresponding extended model which takes into account the random disturbances, control input  $u$  and noisy observations:

$$\begin{aligned}\dot{x} &= x(x - p)(1 - x) + u + \varepsilon x \xi(t) \\ y &= x + \varepsilon \varphi x \eta(t).\end{aligned}\tag{18}$$

Here,  $\xi(t)$  and  $\eta(t)$  are independent standard Gaussian noises:  $E\xi(t) = E\eta(t) = 0$ ,  $E\xi(t)\xi(\tau) = E\eta(t)\eta(\tau) = \delta(t - \tau)$ .

First, consider an influence of the noise on the nontrivial equilibrium  $\bar{x}_2$  for uncontrolled system ( $u = 0$ ). For weak noise, random trajectories starting from this equilibrium demonstrate small-amplitude stochastic oscillations (see Figure 3a for  $\varepsilon = 0.05$ , blue). As the noise intensity increases, the amplitude of these oscillations grows, and the random trajectory intersects the separatrix  $\bar{x}_1$ , falls into the basin of attraction of  $\bar{x}_0$ , and tends to zero (see Figure 3a for  $\varepsilon = 0.2$ , red). It means that the system exhibits the noise-induced extinction. Note that for  $u = 0$ , the stochastic sensitivity of the equilibrium  $\bar{x}_2$  is  $w = \frac{1}{2(1-p)}$ . In what follows, we put  $p = 0.5$ . For  $p = 0.5$ , we have  $w = 1$ .

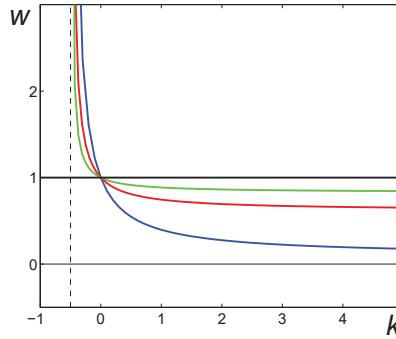
To stabilize this equilibrium and avoid the noise-induced extinction, we have to reduce the stochastic sensitivity. Controlling the stochastic sensitivity is provided by the corresponding dynamic regulator. In the considered case, the regulator is composed from the feedback and the filter

$$u = -k(\hat{x} - \bar{x}_2), \quad \dot{\hat{x}} = \hat{x}(\hat{x} - p)(1 - \hat{x}) + u + l(y - \hat{x}).\tag{19}$$

We will use the optimal Kalman–Bucy filter (15) with

$$l_* = -0.5 + \sqrt{0.25 + \frac{1}{\varphi^2}}.$$

For this  $l = l_*$ , the stochastic sensitivity  $w(k) = w_{11}(k, l_*)$  is the function of  $k$ . In Figure 4, plots of  $w(k)$  for three values



**FIGURE 4.** Plots of  $w(k)$  for  $l = l_*$  for the stochastic population system with regulator for  $\varphi = 2$  (green),  $\varphi = 1$  (red),  $\varphi = 0.1$  (blue). Thick black line mark the sensitivity  $w = 1$  for stochastic population system without control

of the parameter  $\varphi$  are presented. As one can see, using the regulator (19) with  $l = l_*$  and  $k > 0$ , we can decrease the stochastic sensitivity. For example, for  $\varphi = 0.1$ , the feedback with  $k = 4$  gives us  $w \approx 0.2$ . This is five times lesser than the sensitivity  $w = 1$  in uncontrolled system. In Figure 3b, time series of the controlled system with  $l_* = 9.5125$ ,  $k = 4$  are plotted for  $\varepsilon = 0.05$  (blue) and  $\varepsilon = 0.2$  (red).

As one can see, a decrease of the stochastic sensitivity by the dynamic regulator allows us to keep the stochastic trajectories in the basin of attraction of the equilibrium  $\bar{x}$ , and consequently, to avoid the noise-induced extinction.

## CONCLUSION

The control of stochastic systems with the incomplete information is a challenging problem of science and engineering. In the present paper, we considered dynamical systems in which system states are known limitedly, and data

contain random errors. Our control approach is based on the synthesis of the required stochastic sensitivity for the equilibrium. Abilities of the dynamic regulators were studied. The effectiveness of the proposed theoretical approach was demonstrated on the example of the stabilization of the population density in the model with Allee effect.

## REFERENCES

- [1] H. J. Kushner, *Stochastic Stability and Control* (Academic Press, New York, 1967).
- [2] K. J. Astrom, *Introduction to the Stochastic Control Theory* (Academic Press, New York, 1970).
- [3] W. Wonham, *Random Differential Equations in Control Theory* (Academic Press, New York, 1970).
- [4] W. H. Fleming and R. W. Rishel, *Deterministic and Stochastic Optimal Control* (Springer, New York, 1975).
- [5] L. Guo and H. Wang, *Stochastic Distribution Control System Design: a Convex Optimization Approach* (Springer-Verlag, New York, 2010).
- [6] M. Athans (Ed), Special issue on the Linear-Quadratic-Gaussian problem, *IEEE Trans. Autom. Control* **16**(6), (1971).
- [7] W. Wonham (1968) *SIAM J. Control* **6**, 312–326.
- [8] T. Georgiou and A. Lindquist (2013) *IEEE Trans. Autom. Control* **58** 2481–2494.
- [9] J.-Q. Sun, *Stochastic Dynamics and Control* (Elsevier, Amsterdam, 2006).
- [10] J. Hu, Z. Wang, B. Shen, and H. Gao (2013) *Int. J. Control* **86**, 650–663.
- [11] W. Horsthemke and R. Lefever, *Noise-Induced Transitions* (Springer, Berlin, 1984).
- [12] L. Gammaitoni, P. Hanggi, P. Jung, and F. Marchesoni (1998) *Rev. Mod. Phys.* **70**, 223–287.
- [13] Y.-C. Lai and T. Tel, *Transient Chaos: Complex Dynamics on Finite Time Scales* (Springer, New York, 2011).
- [14] M. I. Freidlin and A. D. Wentzell, *Random Perturbations of Dynamical Systems* (Springer, New York, 1984).
- [15] G. N. Mil'shtein and L. B. Ryashko (1995) *J. Appl. Maths. Mechs.* **59**, 47–56.
- [16] I. Bashkirtseva and L. Ryashko (2011) *Chaos* **21**, 047514.
- [17] I. Bashkirtseva and L. Ryashko (2011) *Phys. Rev. E* **83**, 061109.
- [18] I. Bashkirtseva, G. Chen, and L. Ryashko (2012) *Commun. Nonlinear Sci. Numer. Simul.* **17**, 3381–3389.
- [19] I. Bashkirtseva, G. Chen, and L. Ryashko (2013) *Eur. Phys. J. B* **86**, 437.
- [20] I. Bashkirtseva and L. Ryashko (2008) *Automat. Rem. Contr.* **69**, 1171–1180.
- [21] B. Shen, Z. Wang, and H. Shu, *Nonlinear Stochastic Systems with Incomplete Information, Filtering and Control* (Springer, Berlin, 2013).
- [22] I. Bashkirtseva, L. Ryashko, and G. Chen (2017) *J. of the Frankl. Inst.* **354**, 1658–1672.
- [23] I. Bashkirtseva (2017) *Int. J. Contr.*, <http://dx.doi.org/10.1080/00207179.2016.1216608>.
- [24] W. Wonham, *Linear Multivariable Control: A Geometric Approach* (Springer-Verlag, Berlin, 1979).
- [25] W. C. Allee, *Animal Aggregations, a Study in General Sociology* (University of Chicago Press, Chicago, Illinois, 1931).
- [26] B. Dennis (1989) *Nat. Resources Modeling* **3**, 481–538.